Single-Variable Substitution for Substructural Theories

Sanjiv Ranchod

(joint work with Marcelo Fiore)

University of Cambridge

École Polytechnique, France

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Simply-typed single-sorted syntax

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Some (free) symmetric monoidal category
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A presheaf P in $\mathcal{C}$ abstractly models syntax
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$\mathcal{C}$ is symmetric monoidal closed with Day convolution $\otimes$

The Cartesian Case

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Goal: Define a structure on a presheaf P in $\mathcal{F}$ which captures substitution for the syntax P.

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V is the cartesian syntax with only variables as terms

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$(\mathcal{F}, \circ, V)$ monoidal closed category

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$(P, \mu : P \circ P \to P, \eta : V \to P)$

μ performs substitution

η specifies variables

unit laws express substituting with/into variables

associativity law expresses the substitution lemma

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Fiore-Plotkin-Turi (1999): $\mathrm{Mon}(\mathcal{F}) \cong \mathbf{Cln} \cong \mathbf{Law}$

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$$\mathfrak{F}: \qquad I \longrightarrow C \longleftarrow C, C \qquad \text{commutative monoid}$$

Operations correspond to structural rules:

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This universal property induces the substitution tensor on $\mathcal{F}$

Linear, Affine and Relevant Syntax

Linear Case: Exchange

$\mathfrak{B}$: L no equations

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Relevant Case: Contraction and Exchange

$\mathfrak{S}:$ $C \longleftarrow C, C$ commutative semigroup

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$\mathcal{S} = \mathbf{Set}^{\mathbb{S}}$

Fiore-R.: $\mathrm{Mon}(\mathcal{S}) \cong \mathbf{RelOp}$

Single-Variable Substitution

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Syntactic Cartesian Substitution:

$$\frac{x_1, \ldots, x_{n+1} \vdash t \qquad x_1, \ldots, x_n \vdash u}{x_1, \ldots, x_n \vdash t[x_{n+1} := u]}$$

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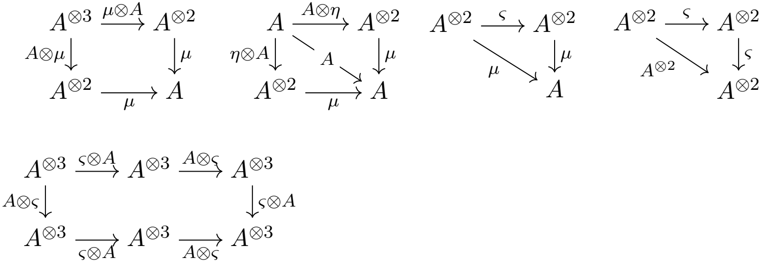

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Operations directly correspond to structural rules

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The endofunctor $- + 1 : \mathbb{F} \to \mathbb{F}$ is a symmetric monad:

$$- + c : - + 2 \to - + 1$$
$$- + w : - + 0 \to - + 1$$
$$- + s : - + 2 \to - + 2$$

Context Extension

This can be lifted to the presheaf category $\mathcal{F}$:

$$\begin{array}{ccc} \mathbb{F}^{\mathrm{op}} & \xrightarrow{\mathcal{Y}} & \mathcal{F} \\ {\scriptstyle (-+1)^{\mathrm{op}}}\downarrow & & \uparrow{\scriptstyle \delta} \\ \mathbb{F}^{\mathrm{op}} & \xrightarrow{\mathcal{Y}} & \mathcal{F} \end{array}$$

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δ is a strong symmetric monad:

$\mathbf{str} : \delta(P) \times Q \to \delta(P \times Q)$
Strength respects the symmetric monad structure

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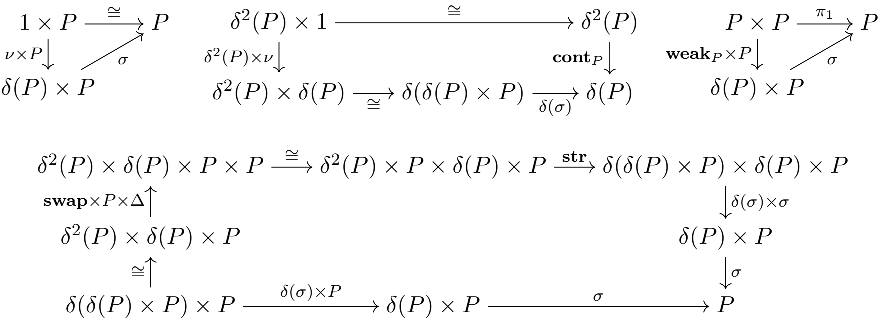

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General Approach

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General Approach

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Corresponding **MEqP**: $\bar{\mathfrak{C}}$

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Category of Contexts: $\mathbb{C} = \mathrm{Th}(\mathfrak{C}) = \mathrm{Th}(\bar{\mathfrak{C}})$

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Model of $\bar{\mathfrak{C}}$ in Endo($\mathbb{C}$): $(- + 1)$

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Model of $\bar{\mathfrak{C}}$ in Endo($\mathcal{C}$): $\delta \rightsquigarrow$ Yoneda restriction of $- + 1$

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Presheaf of Variables: $V = \mathcal{Y}(1)$

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δ has $\otimes$-strength: $- \otimes V \dashv \delta \quad \rightsquigarrow \quad \delta \cong - \multimap V$

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Substitution Algebra: $(P, \sigma : \delta(P) \otimes P \to P, \nu : I \to \delta(P))$

But what about the axioms?

The Linear Case

Monoidal Presentation: Symmetric Object (A, ς) with

$$\begin{array}{ccc} A^{\otimes 2} & \xrightarrow{\varsigma} & A^{\otimes 2} \\ & \searrow^{A^{\otimes 2}} & \downarrow{\scriptstyle \varsigma} \\ & & A^{\otimes 2} \end{array} \qquad \begin{array}{ccccc} A^{\otimes 3} & \xrightarrow{\varsigma\otimes A} & A^{\otimes 3} & \xrightarrow{A\otimes\varsigma} & A^{\otimes 3} \\ {\scriptstyle A\otimes\varsigma}\downarrow & & & & \downarrow{\scriptstyle \varsigma\otimes A} \\ A^{\otimes 3} & \xrightarrow[\varsigma\otimes A]{} & A^{\otimes 3} & \xrightarrow[A\otimes\varsigma]{} & A^{\otimes 3} \end{array}$$

A symmetric endofunctor is a symmetric object in a category of endofunctors

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Monoidal Presentation: Symmetric Object (A, ς) with

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& \searrow^{A^{\otimes 2}} & \downarrow \varsigma \\
& & A^{\otimes 2}
\end{array}
\qquad
\begin{array}{ccccc}
A^{\otimes 3} & \xrightarrow{\varsigma \otimes A} & A^{\otimes 3} & \xrightarrow{A \otimes \varsigma} & A^{\otimes 3} \\
{\scriptstyle A \otimes \varsigma} \downarrow & & & & \downarrow {\scriptstyle \varsigma \otimes A} \\
A^{\otimes 3} & \xrightarrow[\varsigma \otimes A]{} & A^{\otimes 3} & \xrightarrow[A \otimes \varsigma]{} & A^{\otimes 3}
\end{array}
$$

A symmetric endofunctor is a symmetric object in a category of endofunctors

$\mathbb{B}$ is the free monoidal category on a symmetric object $(1, s : 2 \to 2)$

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Monoidal Presentation: Symmetric Object (A, ς) with

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{\scriptstyle A\otimes\varsigma}\downarrow & & & & \downarrow{\scriptstyle \varsigma\otimes A} \\
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\end{array}
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On $\mathbb{B}$: $(- + 1, - + s)$ is a symmetric endofunctor

On $\mathcal{B}$: $(\delta, \mathbf{swap})$ is a strong symmetric endofunctor

Linear Substitution Algebra

A linear substitution algebra is a triple (P, σ, ν) such that:

$$
\begin{array}{ccc}
I \otimes P & \xrightarrow{\cong} & P \\
{\scriptstyle \nu \otimes P}\downarrow & \nearrow_{\sigma} & \\
\delta(P) \otimes P & &
\end{array}
\qquad
\begin{array}{cccc}
\delta(P) \otimes I & & \xrightarrow{\cong} & \delta(P) \\
{\scriptstyle \delta(P) \otimes \nu}\downarrow & & & \nearrow_{\delta(\sigma)} \\
\delta(P) \otimes \delta(P) & \xrightarrow{\mathbf{str}'} & \delta(\delta(P) \otimes P) &
\end{array}
$$

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A linear substitution algebra is a triple (P, σ, ν) such that:

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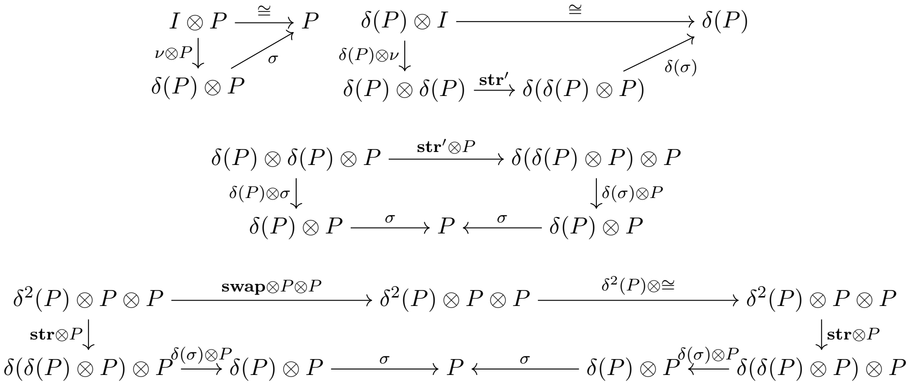
$$\begin{array}{ccccc} \delta(P) \otimes \delta(P) \otimes P & & \xrightarrow{\mathbf{str}' \otimes P} & & \delta(\delta(P) \otimes P) \otimes P \\ {\scriptstyle \delta(P)\otimes\sigma}\downarrow & & & & \downarrow{\scriptstyle \delta(\sigma)\otimes P} \\ \delta(P) \otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P) \otimes P \end{array}$$

$$\begin{array}{ccccccccc} \delta^2(P) \otimes P \otimes P & & & \xrightarrow{\mathbf{swap} \otimes P \otimes P} & \delta^2(P) \otimes P \otimes P & \xrightarrow{\delta^2(P) \otimes \cong} & & & \delta^2(P) \otimes P \otimes P \\ {\scriptstyle \mathbf{str}\otimes P}\downarrow & & & & & & & & \downarrow{\scriptstyle \mathbf{str}\otimes P} \\ \delta(\delta(P) \otimes P) \otimes P & \xrightarrow{\delta(\sigma)\otimes P} & \delta(P) \otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P) \otimes P & \xleftarrow{\delta(\sigma)\otimes P} & \delta(\delta(P) \otimes P) \otimes P \end{array}$$

Linear Substitution Algebra

A linear substitution algebra is a triple (P, σ, ν) such that:



$$\mathbf{LSubstAlg} \cong \mathrm{Mon}(\mathcal{B})$$

The Affine Case

Monoidal Presentation: Symmetric Pointed Object (A, η, ς) with

$$
\begin{array}{ccc}
A^{\otimes 2} & \xrightarrow{\varsigma} & A^{\otimes 2} \\
& {\scriptstyle A^{\otimes 2}} \searrow & \downarrow{\scriptstyle \varsigma} \\
& & A^{\otimes 2}
\end{array}
\qquad
\begin{array}{ccccc}
A^{\otimes 3} & \xrightarrow{\varsigma \otimes A} & A^{\otimes 3} & \xrightarrow{A \otimes \varsigma} & A^{\otimes 3} \\
{\scriptstyle A \otimes \varsigma}\downarrow & & & & \downarrow{\scriptstyle \varsigma \otimes A} \\
A^{\otimes 3} & \xrightarrow[\varsigma \otimes A]{} & A^{\otimes 3} & \xrightarrow[A \otimes \varsigma]{} & A^{\otimes 3}
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\eta \otimes A} & A^{\otimes 2} \\
& {\scriptstyle A \otimes \eta} \searrow & \downarrow{\scriptstyle \varsigma} \\
& & A^{\otimes 2}
\end{array}
$$

A symmetric pointed endofunctor is a symmetric pointed object in a category of endofunctors

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{\scriptstyle A\otimes\varsigma}\downarrow & & & & \downarrow{\scriptstyle \varsigma\otimes A} \\
A^{\otimes 3} & \xrightarrow[\varsigma\otimes A]{} & A^{\otimes 3} & \xrightarrow[A\otimes\varsigma]{} & A^{\otimes 3}
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A & \xrightarrow{\eta\otimes A} & A^{\otimes 2} \\
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$\mathbb{I}$ is the free monoidal category on a symmetric pointed object $(1, w : 0 \to 1, s : 2 \to 2)$

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{\scriptstyle A \otimes \varsigma}\downarrow & & & & \downarrow{\scriptstyle \varsigma \otimes A} \\
A^{\otimes 3} & \xrightarrow[\varsigma \otimes A]{} & A^{\otimes 3} & \xrightarrow[A \otimes \varsigma]{} & A^{\otimes 3}
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\eta \otimes A} & A^{\otimes 2} \\
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$$

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On $\mathbb{I}$: $(- + 1, - + w, - + s)$ is a symmetric pointed endofunctor
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Affine Substitution Algebra

An affine substitution algebra is a triple (P, σ, ν) such that:

$$
\begin{array}{ccc}
I \otimes P & \xrightarrow{\cong} & P \\
{\scriptstyle \nu \otimes P}\downarrow & \nearrow{\scriptstyle \sigma} & \\
\delta(P) \otimes P & &
\end{array}
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\begin{array}{cccc}
\delta(P) \otimes I & & \xrightarrow{\cong} & \delta(P) \\
{\scriptstyle \delta(P) \otimes \nu}\downarrow & & \nearrow{\scriptstyle \delta(\sigma)} & \\
\delta(P) \otimes \delta(P) & \xrightarrow{\mathbf{str}'} & \delta(\delta(P) \otimes P) &
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\delta(P) \otimes \delta(P) & \xrightarrow{\mathbf{str'}} & \delta(\delta(P) \otimes P) & &
\end{array}
\qquad
\begin{array}{ccc}
P \otimes P & \xrightarrow{\pi_1} & P \\
{\scriptstyle \mathbf{weak} \otimes P}\downarrow & \nearrow{\scriptstyle \sigma} & \\
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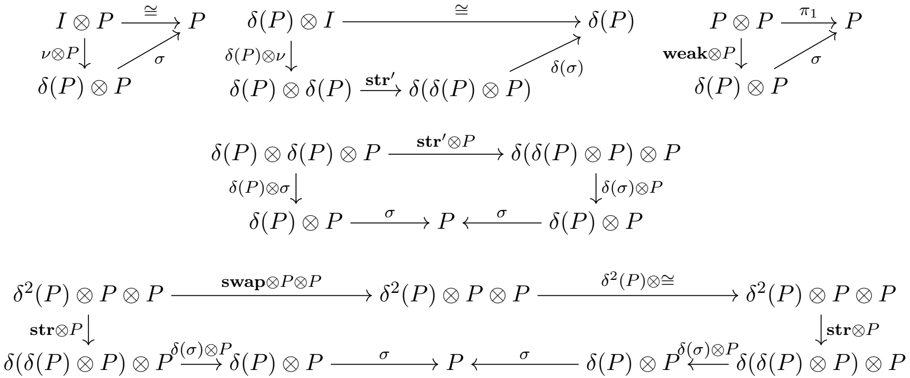

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\delta(P) \otimes \delta(P) & \xrightarrow{\mathbf{str}'} & \delta(\delta(P) \otimes P)
\end{array}
\qquad
\begin{array}{ccc}
P \otimes P & \xrightarrow{\pi_1} & P \\
{\scriptstyle \mathbf{weak} \otimes P}\downarrow & \nearrow_{\sigma} & \\
\delta(P) \otimes P & &
\end{array}
$$

$$
\begin{array}{ccccc}
\delta(P) \otimes \delta(P) \otimes P & & \xrightarrow{\mathbf{str}' \otimes P} & & \delta(\delta(P) \otimes P) \otimes P \\
{\scriptstyle \delta(P) \otimes \sigma}\downarrow & & & & \downarrow{\scriptstyle \delta(\sigma) \otimes P} \\
\delta(P) \otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P) \otimes P
\end{array}
$$

$$
\begin{array}{ccccccccc}
\delta^2(P) \otimes P \otimes P & & & \xrightarrow{\mathbf{swap} \otimes P \otimes P} & \delta^2(P) \otimes P \otimes P & \xrightarrow{\delta^2(P) \otimes \cong} & & & \delta^2(P) \otimes P \otimes P \\
{\scriptstyle \mathbf{str} \otimes P}\downarrow & & & & & & & & \downarrow{\scriptstyle \mathbf{str} \otimes P} \\
\delta(\delta(P) \otimes P) \otimes P & \xrightarrow{\delta(\sigma) \otimes P} & \delta(P) \otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P) \otimes P & \xleftarrow{\delta(\sigma) \otimes P} & \delta(\delta(P) \otimes P) \otimes P
\end{array}
$$

$$\mathbf{ASubstAlg} \cong \mathrm{Mon}(\mathcal{I})$$

The Relevant Case

Monoidal Presentation: Symmetric Semigroup (A, μ, ς) with

$$\begin{array}{ccc} A^{\otimes 3} & \xrightarrow{\mu \otimes A} & A^{\otimes 2} \\ {\scriptstyle A\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A^{\otimes 2} & \xrightarrow[\mu]{} & A \end{array} \qquad \begin{array}{ccc} A^{\otimes 2} & \xrightarrow{\varsigma} & A^{\otimes 2} \\ & {\scriptstyle \mu}\searrow & \downarrow{\scriptstyle \mu} \\ & & A \end{array} \qquad \begin{array}{ccccc} A^{\otimes 3} & \xrightarrow{\varsigma \otimes A} & A^{\otimes 3} & \xrightarrow{A \otimes \varsigma} & A^{\otimes 3} \\ {\scriptstyle A\otimes\mu}\downarrow & & & & \downarrow{\scriptstyle \mu\otimes A} \\ A^{\otimes 2} & & \xrightarrow[\varsigma]{} & & A^{\otimes 2} \end{array}$$

$$\begin{array}{ccc} A^{\otimes 2} & \xrightarrow{\varsigma} & A^{\otimes 2} \\ & {\scriptstyle A^{\otimes 2}}\searrow & \downarrow{\scriptstyle \varsigma} \\ & & A^{\otimes 2} \end{array} \qquad \begin{array}{ccccc} A^{\otimes 3} & \xrightarrow{\varsigma \otimes A} & A^{\otimes 3} & \xrightarrow{A \otimes \varsigma} & A^{\otimes 3} \\ {\scriptstyle A\otimes\varsigma}\downarrow & & & & \downarrow{\scriptstyle \varsigma\otimes A} \\ A^{\otimes 3} & \xrightarrow[\varsigma\otimes A]{} & A^{\otimes 3} & \xrightarrow[A\otimes\varsigma]{} & A^{\otimes 3} \end{array}$$

A symmetric multiplicative endofunctor is a symmetric semigroup in a category of endofunctors

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$$
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A^{\otimes 3} & \xrightarrow{\mu \otimes A} & A^{\otimes 2} \\
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A^{\otimes 2} & \xrightarrow[\mu]{} & A
\end{array}
\qquad
\begin{array}{ccc}
A^{\otimes 2} & \xrightarrow{\varsigma} & A^{\otimes 2} \\
& {\scriptstyle \mu}\searrow & \downarrow{\scriptstyle \mu} \\
& & A
\end{array}
\qquad
\begin{array}{ccccc}
A^{\otimes 3} & \xrightarrow{\varsigma \otimes A} & A^{\otimes 3} & \xrightarrow{A \otimes \varsigma} & A^{\otimes 3} \\
{\scriptstyle A\otimes\mu}\downarrow & & & & \downarrow{\scriptstyle \mu\otimes A} \\
A^{\otimes 2} & & \xrightarrow[\varsigma]{} & & A^{\otimes 2}
\end{array}
$$

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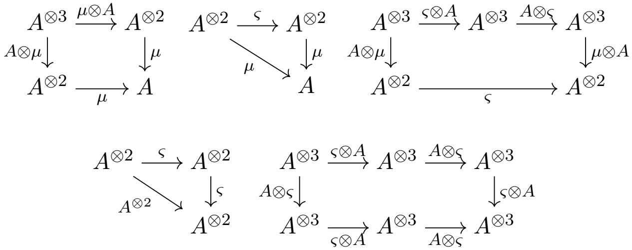
A symmetric multiplicative endofunctor is a symmetric semigroup in a category of endofunctors

$\mathbb{S}$ is the free monoidal category on a symmetric semigroup $(1, c : 2 \to 1, s : 2 \to 2)$

The Relevant Case

Monoidal Presentation: Symmetric Semigroup (A, μ, ς) with



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On $\mathbb{S}$: $(- + 1, - + c, - + s)$ is a symmetric multiplicative endofunctor
On $\mathcal{S}$: $(\delta, \mathbf{cont}, \mathbf{swap})$ is a strong symmetric multiplicative endofunctor

Relevant Substitution Algebra

A relevant substitution algebra is a triple (P, σ, ν) such that:

$$\begin{array}{ccc} I \otimes P & \xrightarrow{\cong} & P \\ {\scriptstyle \nu \otimes P}\downarrow & \nearrow_{\sigma} & \\ \delta(P) \otimes P & & \end{array} \qquad \begin{array}{ccccc} \delta(P) \otimes I & & \xrightarrow{\cong} & & \delta(P) \\ {\scriptstyle \delta(P)\otimes\nu}\downarrow & & & \nearrow_{\delta(\sigma)} & \\ \delta(P) \otimes \delta(P) & \xrightarrow{\mathbf{str}'} & \delta(\delta(P) \otimes P) & & \end{array}$$

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\end{array}
$$

$$
\begin{array}{ccccc}
\delta(P) \otimes \delta(P) \otimes P & & \xrightarrow{\mathbf{str}' \otimes P} & & \delta(\delta(P) \otimes P) \otimes P \\
{\scriptstyle \delta(P) \otimes \sigma}\big\downarrow & & & & \big\downarrow{\scriptstyle \delta(\sigma) \otimes P} \\
\delta(P) \otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P) \otimes P
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\end{array}
$$

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\begin{array}{ccccc}
\delta(P) \otimes \delta(P) \otimes P & & \xrightarrow{\mathbf{str}' \otimes P} & & \delta(\delta(P) \otimes P) \otimes P \\
{\scriptstyle \delta(P)\otimes\sigma}\downarrow & & & & \downarrow{\scriptstyle \delta(\sigma)\otimes P} \\
\delta(P) \otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P) \otimes P
\end{array}
$$

$$
\begin{array}{ccccccc}
\delta^2(P) \otimes P \otimes P & & \xrightarrow{\mathbf{swap} \otimes P \otimes P} & \delta^2(P) \otimes P \otimes P & \xrightarrow{\delta^2(P)\otimes\cong} & & \delta^2(P) \otimes P \otimes P \\
{\scriptstyle \mathbf{str}\otimes P}\downarrow & & & & & & \downarrow{\scriptstyle \mathbf{str}\otimes P} \\
\delta(\delta(P) \otimes P) \otimes P & \xrightarrow{\delta(\sigma)\otimes P} & \delta(P) \otimes P & \xrightarrow{\sigma} P \xleftarrow{\sigma} & \delta(P) \otimes P & \xleftarrow{\delta(\sigma)\otimes P} & \delta(\delta(P) \otimes P) \otimes P
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\end{array}
$$

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\begin{array}{ccccc}
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{\scriptstyle \delta(P)\otimes\sigma}\downarrow & & & & \downarrow{\scriptstyle \delta(\sigma)\otimes P} \\
\delta(P) \otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P) \otimes P
\end{array}
$$

$$
\begin{array}{ccccccccc}
\delta^2(P) \otimes P \otimes P & & & \xrightarrow{\mathbf{swap} \otimes P \otimes P} & \delta^2(P) \otimes P \otimes P & \xrightarrow{\delta^2(P) \otimes \cong} & & & \delta^2(P) \otimes P \otimes P \\
{\scriptstyle \mathbf{str}\otimes P}\downarrow & & & & & & & & \downarrow{\scriptstyle \mathbf{str}\otimes P} \\
\delta(\delta(P) \otimes P) \otimes P & \xrightarrow{\delta(\sigma)\otimes P} & \delta(P) \otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P) \otimes P & \xleftarrow{\delta(\sigma)\otimes P} & \delta(\delta(P) \otimes P) \otimes P
\end{array}
$$

$$
\begin{array}{ccccc}
\delta^2(P) \otimes \delta(P) \otimes P \otimes P & \xrightarrow{\cong} & \delta^2(P) \otimes P \otimes \delta(P) \otimes P & \xrightarrow{\mathbf{str}} & \delta(\delta(P) \otimes P) \otimes \delta(P) \otimes P \\
{\scriptstyle \Delta}\uparrow & & & & \downarrow{\scriptstyle \delta(\sigma)\otimes\sigma} \\
\delta^2(P) \otimes \delta(P) \otimes P & & & & \delta(P) \otimes P \\
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\end{array}
$$

Relevant Substitution Algebra

A relevant substitution algebra is a triple (P, σ, ν) such that:

$$\begin{array}{ccc} I\otimes P & \xrightarrow{\cong} & P \\ {\scriptstyle \nu\otimes P}\downarrow & \nearrow{\scriptstyle\sigma} & \\ \delta(P)\otimes P & & \end{array} \qquad \begin{array}{ccc} \delta(P)\otimes I & \xrightarrow{\cong} & \delta(P) \\ {\scriptstyle\delta(P)\otimes\nu}\downarrow & & \nearrow{\scriptstyle\delta(\sigma)} \\ \delta(P)\otimes\delta(P) & \xrightarrow{\mathbf{str}'} & \delta(\delta(P)\otimes P) \end{array}$$

$$\begin{array}{ccccc} \delta(P)\otimes\delta(P)\otimes P & & \xrightarrow{\mathbf{str}'\otimes P} & & \delta(\delta(P)\otimes P)\otimes P \\ {\scriptstyle\delta(P)\otimes\sigma}\downarrow & & & & \downarrow{\scriptstyle\delta(\sigma)\otimes P} \\ \delta(P)\otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P)\otimes P \end{array}$$

$$\begin{array}{ccccccccc} \delta^2(P)\otimes P\otimes P & & & \xrightarrow{\mathbf{swap}\otimes P\otimes P} & \delta^2(P)\otimes P\otimes P & \xrightarrow{\delta^2(P)\otimes\cong} & & & \delta^2(P)\otimes P\otimes P \\ {\scriptstyle\mathbf{str}\otimes P}\downarrow & & & & & & & & \downarrow{\scriptstyle\mathbf{str}\otimes P} \\ \delta(\delta(P)\otimes P)\otimes P & \xrightarrow{\delta(\sigma)\otimes P} & \delta(P)\otimes P & \xrightarrow{\sigma} & P & \xleftarrow{\sigma} & \delta(P)\otimes P & \xleftarrow{\delta(\sigma)\otimes P} & \delta(\delta(P)\otimes P)\otimes P \end{array}$$

$$\begin{array}{ccccc} \delta^2(P)\otimes\delta(P)\otimes P\otimes P & \xrightarrow{\cong} & \delta^2(P)\otimes P\otimes\delta(P)\otimes P & \xrightarrow{\mathbf{str}} & \delta(\delta(P)\otimes P)\otimes\delta(P)\otimes P \\ {\scriptstyle\Delta}\uparrow & & & & \downarrow{\scriptstyle\delta(\sigma)\otimes\sigma} \\ \delta^2(P)\otimes\delta(P)\otimes P & & & & \delta(P)\otimes P \\ {\scriptstyle\rho\otimes P}\downarrow & & & & \downarrow{\scriptstyle\sigma} \\ \delta(\delta(P)\otimes P)\otimes P & \xrightarrow[\delta(\sigma)\otimes P]{} & \delta(P)\otimes P & \xrightarrow[\sigma]{} & P \end{array}$$

$$\mathbf{RSubstAlg} \cong \mathrm{Mon}(\mathcal{S})$$

Binding Signatures

A binding signature is a pair $\Sigma = (\Omega, \alpha)$ where

Ω is a set of operations

$\alpha : \Omega \to \mathbb{N}^*$ is an arity map

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Second-Order Theories
- Impose equations on the operations
- Requires meta-variables and meta-subtitution
- See Fiore, Mahmoud, Hur, Szamozvancev for cartesian case

Binding for Cartesian Syntax

Given some binding signature $\Sigma = (\Omega, \alpha)$, define

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Does Σ inherit properties of δ?

Strength: Yes!

Symmetric Monad: No $\rightsquigarrow$ But Σ has **swap**

Symmetric Distributive Laws

A symmetric distributive law between a symmetric monad $(T, \mu, \eta, \varsigma)$ and an endofunctor S is

$$\tau : TS \to ST$$

$$\begin{array}{ccc} S & & \\ {\scriptstyle \eta S}\downarrow & \searrow^{S\eta} & \\ TS & \xrightarrow[\tau]{} & ST \end{array} \qquad \begin{array}{ccccc} TTS & \xrightarrow{T\tau} & TST & \xrightarrow{\tau T} & STT \\ {\scriptstyle \mu S}\downarrow & & & & \downarrow{\scriptstyle S\mu} \\ TS & & \xrightarrow[\tau]{} & & ST \end{array} \qquad \begin{array}{ccccc} TTS & \xrightarrow{T\tau} & TST & \xrightarrow{\tau T} & STT \\ {\scriptstyle \varsigma S}\downarrow & & & & \downarrow{\scriptstyle S\varsigma} \\ TTS & \xrightarrow{T\tau} & TST & \xrightarrow{\tau T} & STT \end{array}$$

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For a endofunctor with (cartesian) strength $(F, \mathbf{str})$

- $- \times Q$ is a symmetric comonad
- $\mathbf{str}_{-,Q} : (- \times Q)F \to F(- \times Q)$ is a symmetric codistributive law

Lifting Symmetric Distributive Laws

Let T oplax monoidal symmetric monad and G_1 and G_2 endofunctors and $\psi_1 : TG_1 \to G_1T$ and $\psi_2 : TG_2 \to G_2T$ symmetric distributive laws , then

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Want: A cartesian substitution algebra structure on TV

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Syntactically: Substitution defined recursively

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Universal property of initial algebra = structural recursion

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Given adjoint endofunctors $F \dashv G$ and endofunctors S and S'
with initial S-algebra $\alpha : S(A) \to A$ and S'-algebra $\beta : S'(B) \to B$,
and a natural transformation $\psi : FS \to S'F$
there exists a unique $f : F(A) \to B$ such that

$$\begin{array}{ccc}
S'(F) & \xrightarrow{S'(f)} & S'(B) \\
{\scriptstyle \psi_A}\uparrow & & \\
FS(A) & & \downarrow{\scriptstyle \beta} \\
{\scriptstyle F(\alpha)}\downarrow{\scriptstyle \cong} & & \\
F(A) & \overset{f}{\dashrightarrow} & B
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Note: $\delta(V) \cong \mathcal{Y}(1, - + 1) \cong V + 1$

$$\beta : \delta(V) \times TV \cong V \times TV + 1 \times TV \xrightarrow{[\eta_V \pi_1, \pi_2]} TV$$

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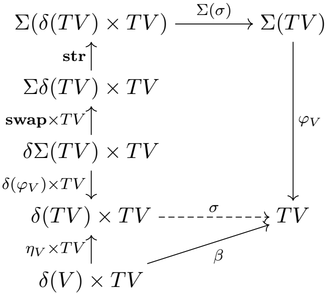

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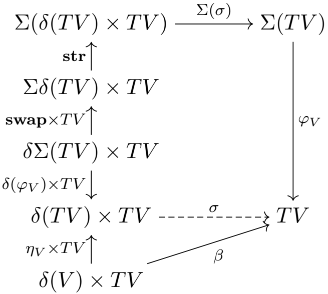

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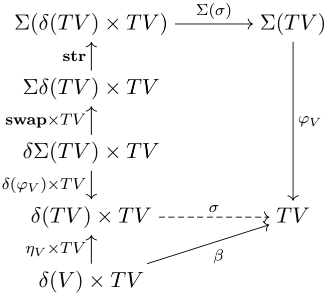

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Binding Signature Endofunctor on $\mathcal{B}$:

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Leibniz Rule: $\mathcal{L} : \delta(P \otimes Q) \cong \delta(P) \otimes Q + P \otimes \delta(Q)$

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By definition: $\mathbf{swap} : \delta\Sigma(P) \cong \Sigma^\dagger(P, \delta(P))$

$\Sigma^\dagger$ linear in Q: $\mathbf{str} : \Sigma^\dagger(P, Q) \otimes R \to \Sigma^\dagger(P, Q \otimes R)$

Every $h : \Sigma(P) \to P$ induces $h^\dagger : \Sigma^\dagger(P) = \Sigma^\dagger(P, P) \to P$

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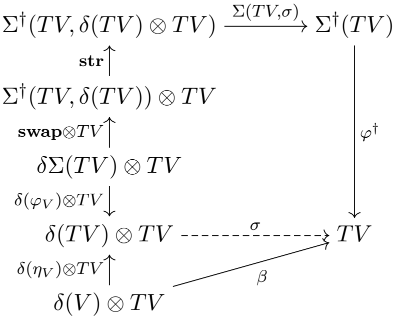
Note: $\delta(V) \cong I$

$\beta : \delta(V) \otimes TV \to TV$

Substitution:



Substitution for Linear Abstract Syntax

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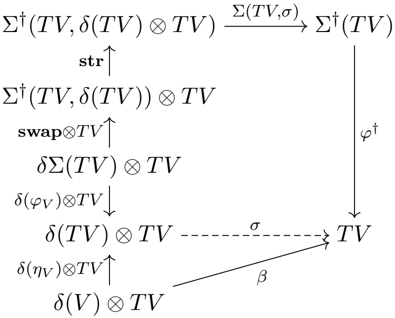

(TV, σ, ν) is a linear substitution algebra

$(TV, \sigma, \nu, \varphi_V)$ is initial Σ-**LSubstAlg**

Affine and Relevant Cases

Binding Signature Endofunctor: $\Sigma(P) = \coprod_{\omega \in \Omega} \bigotimes_{i \in [k]} \delta^{n_i}(P)$

Abstract Syntax: $[\eta_V, \varphi_V] : V + \Sigma(TV) \to TV$

Affine and Relevant Cases

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Leibniz Rules:

Affine $\mathcal{I}$: $\delta(P \otimes Q) \cong \delta(P) \otimes Q + P \otimes \delta(Q) + P \otimes Q$

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Derived Signature Endofunctors: Appropriately defined in each case

By definition: $\mathbf{swap} : \delta\Sigma(P) \cong \Sigma^\dagger(P, \delta(P))$

Day convolution structure: $\mathbf{str} : \Sigma^\dagger(P, Q) \otimes R \to \Sigma^\dagger(P, Q \otimes R)$

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Substitution Algebras: Induced as before

Affine $\mathcal{I}$: $(TV, \sigma, \nu, \varphi_V)$ initial Σ-**ASubstAlg**

Relevant $\mathcal{S}$: $(TV, \sigma, \nu, \varphi_V)$ initial Σ-**RSubstAlg**

Towards Uniform Axiomatisation

Substitution: $\sigma : \delta(P) \otimes P \to P$

Substitution Signature: $\Sigma_{\text{sub}}(P) = \delta(P) \otimes P$

$\Sigma^{\dagger}_{\text{sub}}$ has **swap** and **str**

We can use this to rewrite the axioms of a substitution algebra

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A substitution algebra is a triple $(P, \sigma : \Sigma_{\text{sub}}(P) \to P, \nu : I \to P)$ such that

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\begin{array}{ccc}
I \otimes P & \xrightarrow{\cong} & P \\
{\scriptstyle \nu \otimes P}\downarrow & \nearrow_{\sigma} & \\
\Sigma_{\text{sub}}(P) & &
\end{array}
\qquad
\begin{array}{ccccc}
\delta(P) \otimes I & & \xrightarrow{\cong} & & \delta(P) \\
{\scriptstyle \delta(P)\otimes\nu}\downarrow & & & \nearrow_{\delta(\sigma)} & \\
\delta(P) \otimes \delta(P) & \xrightarrow{\mathbf{str}'} & \delta\Sigma_{\text{sub}}(P) & &
\end{array}
$$

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\begin{array}{ccccccc}
\delta\Sigma_{\text{sub}}(P) \otimes P & \xrightarrow{\mathbf{swap}\otimes P} & \Sigma^{\dagger}_{\text{sub}}(P, \delta(P)) \otimes P & \xrightarrow{\mathbf{str}} & \Sigma^{\dagger}_{\text{sub}}(P, \Sigma_{\text{sub}}(P)) & \xrightarrow{\Sigma^{\dagger}_{\text{sub}}(P,\sigma)} & \Sigma^{\dagger}_{\text{sub}}(P) \\
{\scriptstyle \delta(\sigma)\otimes P}\downarrow & & & & & & \downarrow{\scriptstyle \sigma^{\dagger}} \\
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{\scriptstyle \delta(\sigma)\otimes P}\downarrow & & & & & & \downarrow{\scriptstyle \sigma^{\dagger}} \\
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This works for Linear, Affine and Relevant, but not for Cartesian!