Substructural Abstract Syntax with Variable Binding and Single-Variable Substitution

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Introduction

Focus:

Substructural Syntax with Variable Binding

Substitution

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Substructural Syntax:

Cartesian $\rightsquigarrow$ weakening + contraction + exchange

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Cartesian $\rightsquigarrow$ weakening + contraction + exchange
Linear $\rightsquigarrow$ exchange
Affine $\rightsquigarrow$ weakening + exchange
Relevant $\rightsquigarrow$ contraction + exchange

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Signatures involve operations which may bind variables

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Affine $\rightsquigarrow$ weakening + exchange
Relevant $\rightsquigarrow$ contraction + exchange

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Substitution:

Simultaneous substitution
Capture-avoiding single-variable substitution

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Simultaneous Substitution:

Cartesian: Fiore-Plotkin-Turi (1999)

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Single-Variable Substitution:

Cartesian: Fiore-Plotkin-Turi (1999)
Others: Open

Setting

Category Theoretic "Presheaf Model"

Contexts: (Universal) monoidal category generated by structural rules

Syntax: Covariant presheaves over contexts

$P(\Gamma) = \{\text{terms for syntax } P \text{ in context } \Gamma\}$

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eg. Linear Setting

Symmetric object: $(A, s : A \otimes A \to A \otimes A)$
$s \rightsquigarrow$ exchange

Contexts : $\mathbb{B}$ = free monoidal category over symmetric object

Syntax : $\mathcal{B} = \mathbf{Set}^{\mathbb{B}}$ = combinatorial species

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Context Extension: $\delta \rightsquigarrow$ endofunctor on presheaves

$$\delta(P)(\Gamma) = P(\Gamma + 1)$$

Axiomatisation of Substitution

Data:

Syntax: P

Substitution: $\sigma : \delta(P) \otimes P \to P$

Variables: $\nu : 1 \to \delta(P)$

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Finite equational presentation

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eg. Linear Axioms:

Two Unitor Laws $\rightsquigarrow$ Behaviour of Variables

Two Operad Laws $\rightsquigarrow$ Successive Substitutions

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eg. Linear Axioms:

- Two Unitor Laws $\rightsquigarrow$ Behaviour of Variables
- Two Operad Laws $\rightsquigarrow$ Successive Substitutions
  - equivalently: Extended Substitution Lemma

Abstract Syntax for a Binding Signature

Binding Signature: $\Sigma \rightsquigarrow$ Endofunctor on presheaves

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Abstract Syntax:

$TV \rightsquigarrow$ Free Σ-algebra on V

Fixed point: $TV = \mu X.V + \Sigma(X)$

Representation Independent

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Representation Independent

eg. Linear Lambda Calculus

$\Sigma_\lambda(X) = X \otimes X + \delta(X) \rightsquigarrow$ application and abstraction

Abstract Syntax : $\Lambda = \mu X.V + X \otimes X + \delta(X)$

Substitution for Abstract Syntax

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Cartesian Solution: Fiore-Plotkin-Turi (1999)

Approach for Other Cases:

Show $\delta(TV)$ admits structural recursion by being an initial algebra.

Uniformity Property and Leibniz Isomorphism

Uniformity Property

Given the following situation:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle H}\downarrow & \cong & \downarrow{\scriptstyle H} \\ \mathcal{D} & \xrightarrow[G]{} & \mathcal{D} \end{array}$$

H is a left adjoint and μF exists $\implies H(\mu F) \cong \mu G$

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Leibniz Isomorphism

How to "simplify" $\delta(X \otimes Y)$

Different for each settings

eg. Linear Setting: $\mathcal{L} : \delta(X \otimes Y) \cong \delta(X) \otimes Y + X \otimes \delta(Y)$

Linear Lambda Calculus

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{V+\Sigma_\lambda} & \mathcal{B} \\
\left\downarrow{\scriptstyle \langle \mathrm{Id},\delta \rangle}\right. & & \left\downarrow{\scriptstyle \langle \mathrm{Id},\delta \rangle}\right. \\
\mathcal{B} \times \mathcal{B} & \overset{G}{\dashrightarrow} & \mathcal{B} \times \mathcal{B}
\end{array}
$$

Linear Lambda Calculus

X

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Linear Lambda Calculus

$$X \longmapsto \langle X, \delta(X) \rangle$$

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X & \longmapsto & V + X \otimes X + \delta(X) \\
\big\downarrow & & \big\downarrow \\
\langle X, \delta(X) \rangle & & \langle V+X\otimes X+\delta(X),\ \delta(V+X\otimes X+\delta(X)) \rangle
\end{array}
$$

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\downarrow & & \downarrow \\
 & & \langle V+X\otimes X+\delta(X), \; \delta(V+X\otimes X+\delta(X))\rangle \\
 & & \downarrow \mathcal{L} \; \cong \\
\langle X, \delta(X)\rangle & & \langle V+X\otimes X+\delta(X), \; \delta(V)+\delta(X)\otimes X+X\otimes\delta(X)+\delta\delta(X)\rangle
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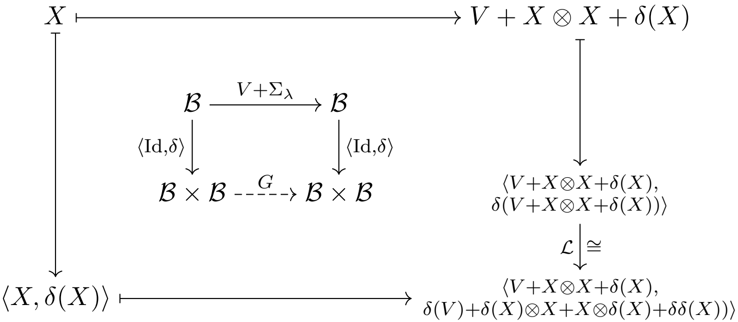
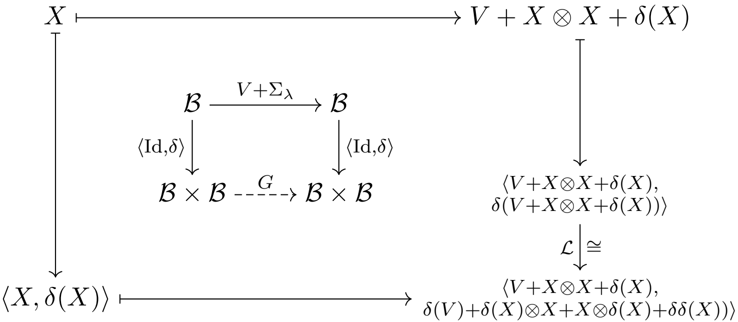
$$G(X, Y) = \langle V + X \otimes X + Y, \delta(V) + Y \otimes X + X \otimes Y + \delta(Y)\rangle$$

Linear Lambda Calculus



$$G(X,Y) = \langle V + X \otimes X + Y, \delta(V) + \underbrace{Y \otimes X + X \otimes Y + \delta(Y)}_{\text{Derived Functor: } \Sigma_\lambda^\dagger(X,Y)} \rangle$$

Linear Lambda Calculus



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Uniformity Property: $\langle \Lambda, \delta(\Lambda) \rangle$ is the fixed point of

$$\begin{cases} X = V + X \otimes X + Y \\ Y = \delta(V) + X \otimes Y + Y \otimes X + \delta(Y) \end{cases}$$

Generalised Structural Recursion

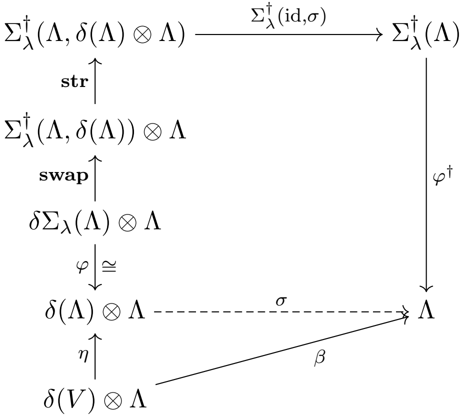
Bird-Paterson (1999): Generalised Structural Recursion

Λ initial $\rightsquigarrow$ admits iterator

$\implies$ $\delta(\Lambda)$ admits generalised iterator $\rightsquigarrow$ corresponds to initiality conditions

Matthes-Uustalu (2003): Useful special case

Substitution for Abstract Syntax



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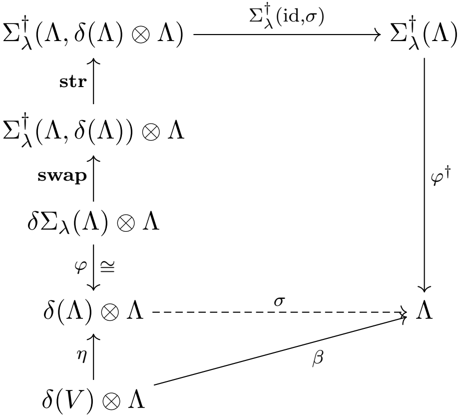

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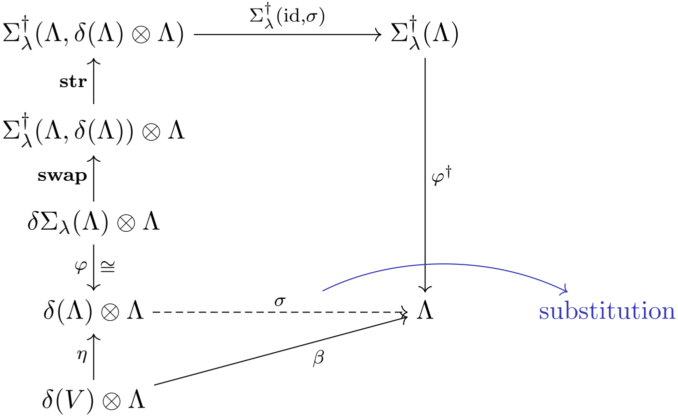

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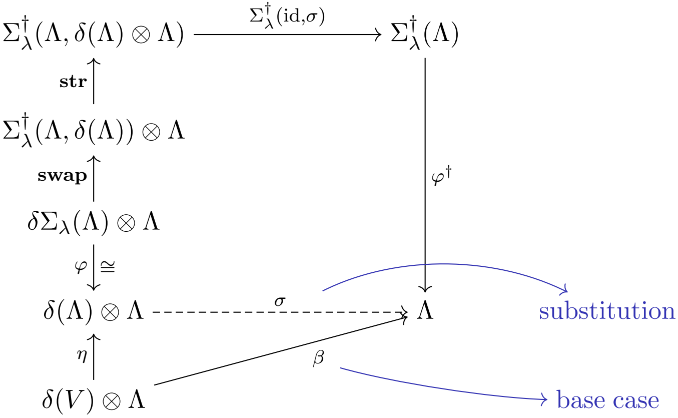

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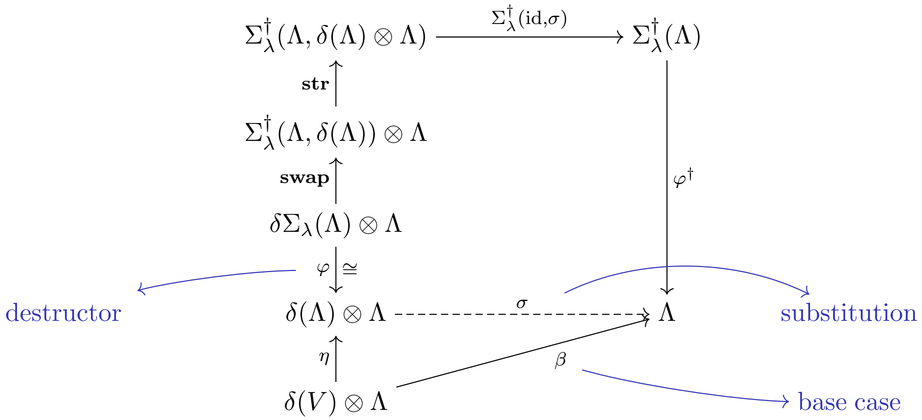

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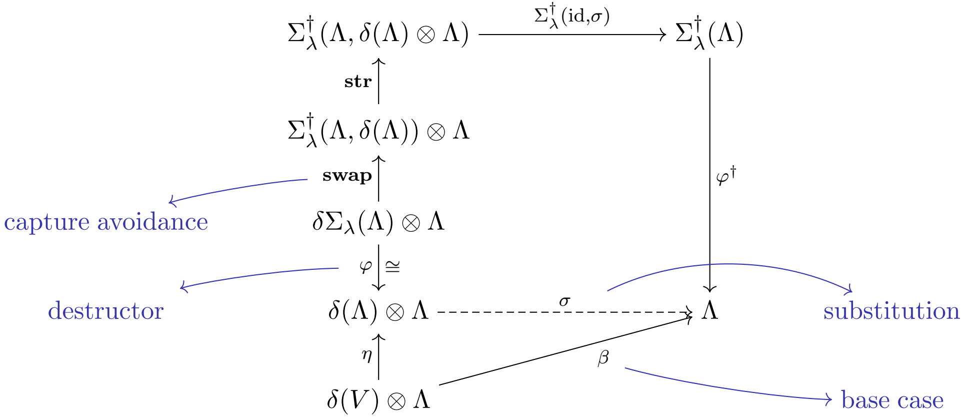

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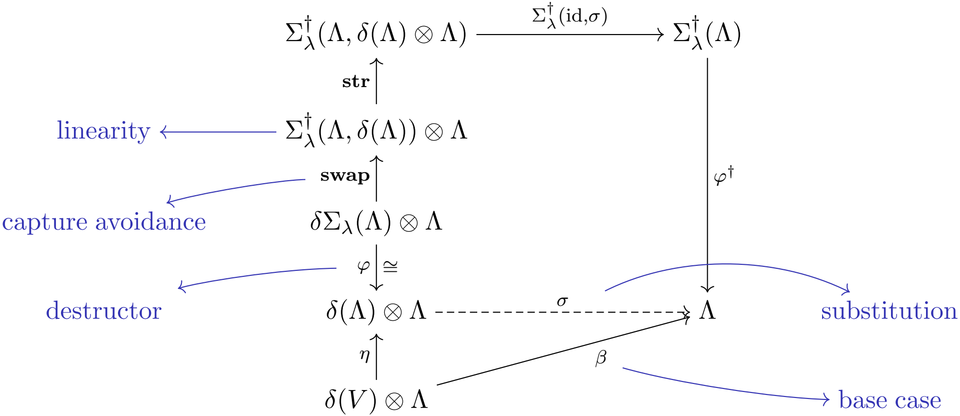

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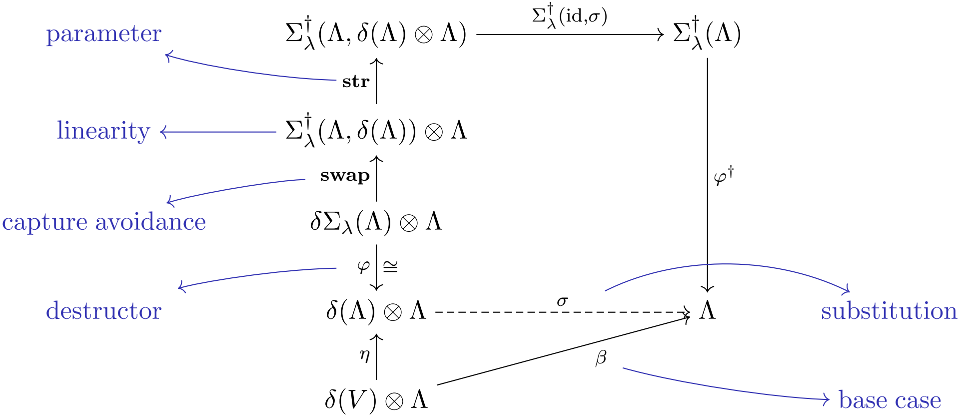

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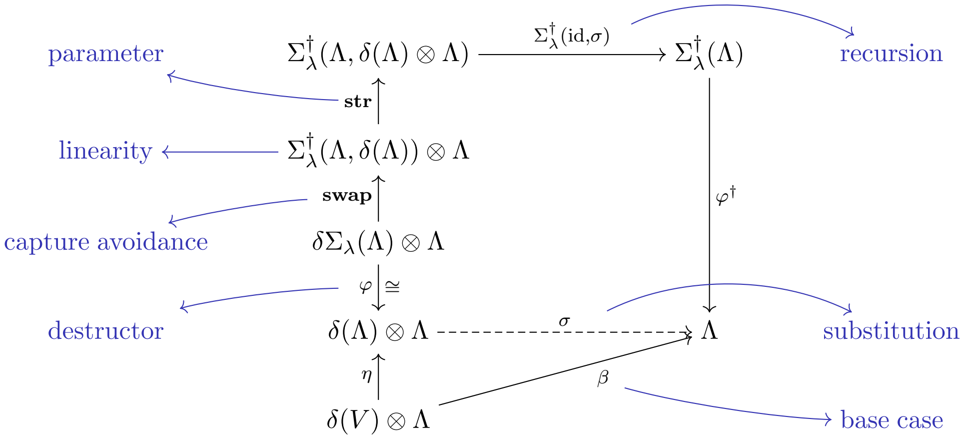

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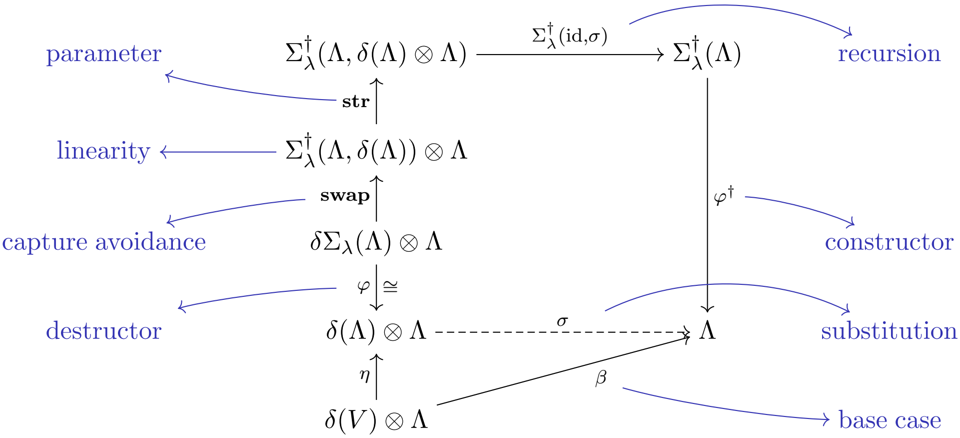

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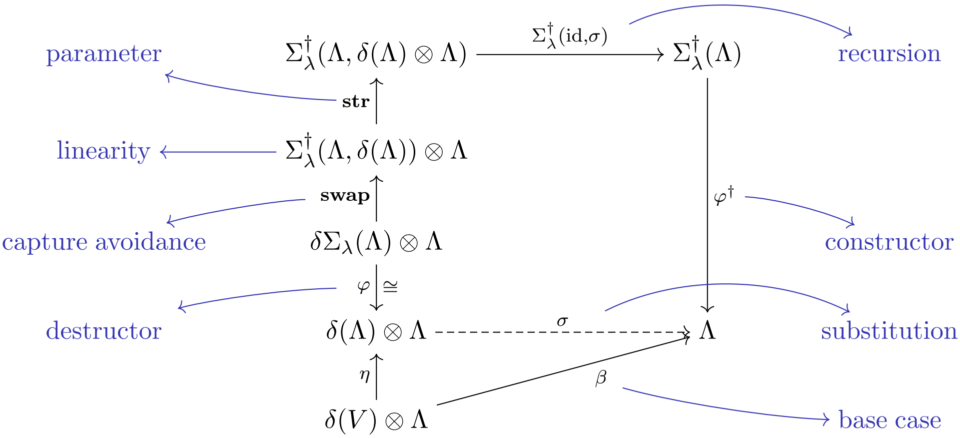

Thm: Λ is the initial Σ_λ-algebra with compatible substitution structure.

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2. Model Binding Signature as Endofunctor Σ
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6. Show Induced Substitution Structure is Universal

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1. Axiomatise Substitution
2. Model Binding Signature as Endofunctor Σ
3. Model Abstract Syntax as Free Algebra TV
4. Uniformity Transfers Universal Property to $\delta(TV)$
5. Induce Substitution Structure on Abstract Syntax
6. Show Induced Substitution Structure is Universal
7. Extract Program for Substitution

Future Work

Second-Order Theories for Linear, Affine and Relevant Settings

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Single-Variable Substitution for Combined Settings

eg. Linear-Cartesian Setting