Substitution for Linear-Cartesian and Full Substructural Theories

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University of Cambridge

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Overview

Substitution for Cartesian and Linear Theories

Concretely

Universally

Using Symmetric Monoidal Theories

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Extending to Other Theories

- Linear-Cartesian Theories
- Full Substructural Theories

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Free-Forgetful Adjunctions

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Other Aspects of the Work

Bicategories

A Broader Class of Theories

Single-Variable Substitution

Cartesian Theories

$$x_1, \ldots, x_n \vdash t$$

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Category of Cartesian Contexts: $\mathbb{F}$

Objects: $n \in \mathbb{N}$

Cartesian Theories

$$x_1, \ldots, x_n \vdash t \qquad \mapsto \qquad x_1, \ldots x_m \vdash t'$$

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$\mathbb{F}$ is the free cocartesian category on one object

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Category for Cartesian Syntax: $\mathcal{F} = \mathbf{Set}^{\mathbb{F}}$

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Syntactic Substitution:

$$\frac{x_1, \ldots, x_n \vdash t \qquad \{x_1, \ldots, x_m \vdash u_i\}_{i \in [n]}}{x_1, \ldots, x_m \vdash t[x_i := u_i]_{i \in [n]}}$$

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Substitution for P: $\qquad P(n) \times P(m)^n \to P(m)$

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$$\begin{tikzcd} V \circ P \arrow[r, "\eta \circ P"] \arrow[dr, "\cong"'] & P \circ P \arrow[d, "\mu"] \\ & P \end{tikzcd}$$

substituting a term into

a variable returns the term

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substituting variables into

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substitution lemma

Fiore-Plotkin-Turi (1999) : $\mathrm{Mon}(\mathcal{F}) \cong$ **Law**

Linear Theories

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Day convolution inherits structural properties of the base tensor

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Presheaf of Variables: $V = \mathcal{Y}(1)$

$(\mathcal{B}, \circ, V)$ is a closed monoidal category

Kelly (2005): $\mathrm{Mon}(\mathcal{B}) \cong \mathbf{SymOp}$

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Symmetric Monoidal Equational Presentations

$$\mathbf{Sig} = (\mathbf{Sorts},\ \mathbf{Op},\ \mathbf{Ar} : \mathbf{Op} \to \mathbf{Sorts}^* \times \mathbf{Sorts}) \qquad \mathbf{Eq}$$

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For $\mathfrak{X}$ in **SMEqP** and $\mathbb{C}$ symmetric monoidal category

Models: $\mathrm{Mod}(\mathfrak{X}, \mathbb{C})$ Theories: $\mathrm{Th}(\mathfrak{X})$

Universal Property: $\mathrm{Mod}(\mathfrak{X}, \mathbb{C}) \cong \mathrm{SM}(\mathrm{Th}(\mathfrak{X}), \mathbb{C})$

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Symmetric Monoidal Equational Presentations

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$$\mathrm{coTh}(\mathfrak{X}) \cong \mathrm{Th}(\mathfrak{X})^{\mathrm{op}}$$

Equational Presentations $\mathfrak{F}$ and $\mathfrak{B}$

$\mathfrak{F}$: $\quad I \longrightarrow C \longleftarrow C, C \quad$ commutative monoid

Models: $\mathrm{Mod}(\mathfrak{F}, \mathbb{C}) = \mathrm{CMon}(\mathbb{C})$ Theory: $\mathrm{Th}(\mathfrak{F}) = \mathbb{F}$

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$$Q \longmapsto Q \longmapsto Q^{\times -}$$

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$\mathfrak{B}$: $\quad L \quad$ no equations

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Equational Presentations $\mathfrak{I}$ and $\mathfrak{S}$

$\mathfrak{I}$: $\quad I \longrightarrow A \quad$ no equations

Models: $\mathrm{Mod}(\mathfrak{I}, \mathbb{C}) = \mathrm{PtOb}(\mathbb{C})$ $\quad$ Theory: $\mathrm{Th}(\mathfrak{I}) = \mathbb{I}$

$$\mathcal{I} \hookrightarrow \mathrm{coPtOb}(\mathcal{I}) = \mathrm{coMod}(\mathfrak{I}, \mathcal{I}) \xrightarrow{\cong} \mathrm{SM}(\mathbb{I}^{\mathrm{op}}, \mathcal{I})$$

$$Q \longmapsto Q \longmapsto Q^{\otimes -}$$

Equational Presentations $\mathfrak{I}$ and $\mathfrak{S}$

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$\mathfrak{S}$: $\quad R \longleftarrow R, R \quad$ commutative semigroup

Models: $\mathrm{Mod}(\mathfrak{S}, \mathbb{C}) = \mathrm{CSGrp}(\mathbb{C})$ Theory: $\mathrm{Th}(\mathfrak{S}) = \mathbb{S}$

$$\begin{array}{ccccc} \mathcal{S} & \hookrightarrow & \mathrm{CcoSGrp}(\mathcal{S}) = \mathrm{coMod}(\mathfrak{S}, \mathcal{S}) & \xrightarrow{\cong} & \mathrm{SM}(\mathbb{S}^{\mathrm{op}}, \mathcal{S}) \\ Q & \longmapsto & Q & \longmapsto & Q^{\otimes -} \end{array}$$

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Terms: $\underbrace{x_1, \ldots, x_n}_{\text{linear}} \; ; \; \underbrace{y_1, \ldots, y_m}_{\text{cartesian}} \vdash t$

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$\mathfrak{L}$: $$\begin{tikzcd} I \arrow[r] & C & C, C \arrow[l] \\ & L \arrow[u] & \end{tikzcd}$$

C commutative monoid

Models: $\mathrm{Mod}(\mathfrak{L}, \mathbb{C}) = \mathbb{C}/U$ where $U : \mathrm{CMon}(\mathbb{C}) \to \mathbb{C}$

Linear-Cartesian Theories

Theory: $\mathrm{Th}(\mathfrak{L}) = \mathbb{L}$

Objects: $(\ell, c) \in \mathbb{N}^2$

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Morphisms:

$$(\ell_L + \ell_C, c) \xrightarrow{f} (\ell', c')$$

$$f_L : [\ell_L] \to [\ell'] \qquad \text{bijection}$$

$$f_C : [\ell_C + c] \to [c'] \qquad \text{function}$$

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$$\begin{array}{ccc} & \mathbb{F} \xrightarrow[\iota]{} \mathbb{L} & \\ n & \longmapsto & (0, n) \end{array}$$

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$$\begin{tikzcd}
\ell + c & (\ell, c) \arrow[l, mapsto] \\
\mathbb{F} \arrow[r, shift right, "\iota"'] \arrow[r, phantom, "\perp"] & \mathbb{L} \arrow[l, shift right, "s"'] \\
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(\ell_L + \ell_C, c) \arrow[rr, "f"] \arrow[dr] & & (\ell', c') \\
& (\ell_L, \ell_C + c) \arrow[ur, "{(f_L, f_C)}"'] &
\end{tikzcd}$$

$f_L : [\ell_L] \to [\ell']$ bijection

$f_C : [\ell_C + c] \to [c']$ function

$$\begin{tikzcd}
\ell + c & & (\ell, c) \arrow[ll, mapsto] \arrow[dd] \\
\mathbb{F} \arrow[r, shift right, "\iota"'] & \mathbb{L} \arrow[l, shift right, "s"'] \arrow[l, phantom, "\perp"] & \\
n \arrow[rr, mapsto] & & (0, n)
\end{tikzcd}$$

Linear-Cartesian Theories

Theory: $\mathrm{Th}(\mathfrak{L}) = \mathbb{L}$

Objects: $(\ell, c) \in \mathbb{N}^2$

Morphisms:

$$\begin{tikzcd}
(\ell_L + \ell_C, c) \arrow[rr, "f"] \arrow[dr] & & (\ell', c') \\
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\end{tikzcd}$$

$$f_L : [\ell_L] \to [\ell'] \qquad \text{bijection}$$

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$$\begin{tikzcd}
\ell + c \arrow[d] & & (\ell, c) \arrow[ll, mapsto] \arrow[d] \\
& \mathbb{F} \arrow[r, shift right, "\iota"'] \arrow[r, phantom, "\perp"] & \mathbb{L} \arrow[l, shift right, "s"'] \\
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n \arrow[rr, mapsto] & & (0, n)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F} \arrow[r, shift right, "\iota"'] \arrow[r, phantom, "\perp"] & \mathbb{L} \arrow[l, shift right, "s"'] \arrow[loop right, "S"]
\end{tikzcd}$$

Linear-Cartesian Theories

$$\begin{tikzcd}
\mathbb{F}^{\mathrm{op}} \arrow[d, shift right=1.5ex, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] \\
\mathbb{L}^{\mathrm{op}} \arrow[u, shift right=1.5ex, "s^{\mathrm{op}}"'] \arrow[loop below, "S^{\mathrm{op}}"]
\end{tikzcd}$$

Linear-Cartesian Theories

$$
\begin{tikzcd}
\mathbb{F}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[d, shift right=3, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{F} \arrow[d, shift right=8, "\iota_!"'] \arrow[d, shift left=8, "s^*"] \\
\mathbb{L}^{\mathrm{op}} \arrow[u, shift right=3, "s^{\mathrm{op}}"'] \arrow[r, "\mathcal{Y}"] \arrow[loop below, "S^{\mathrm{op}}"] & \mathcal{L} \arrow[u, "\dashv s_! \dashv" description]
\end{tikzcd}
$$

Linear-Cartesian Theories

$$\begin{tikzcd}
\mathbb{F}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[d, shift right=3, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{F} \arrow[d, shift right=5, "\iota_!"'] \arrow[d, shift left=5, "s^*"] \\
\mathbb{L}^{\mathrm{op}} \arrow[u, shift right=3, "s^{\mathrm{op}}"'] \arrow[r, "\mathcal{Y}"] \arrow[loop below, "S^{\mathrm{op}}"] & \mathcal{L} \arrow[u, "\dashv s_! \dashv" description] \arrow[loop below, "\mathbf{CCr}"]
\end{tikzcd}$$

Cartesian Core: $\mathbf{CCr}(Q)(\ell, c) = \begin{cases} Q(0, c) & \ell = 0 \\ \emptyset & \text{otherwise} \end{cases}$

Linear-Cartesian Theories

$$
\begin{tikzcd}
\mathbb{F}^{\mathrm{op}} \arrow[rr, "\mathcal{Y}"] \arrow[d, shift right, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] & & \mathcal{F} \arrow[d, shift right=2, "\iota_!"'] \arrow[d, shift left=2, "s^*"] \arrow[d, phantom, "\dashv s_! \dashv"] \\
\mathbb{L}^{\mathrm{op}} \arrow[rr, "\mathcal{Y}"] \arrow[u, shift right, "s^{\mathrm{op}}"'] \arrow[loop below, "S^{\mathrm{op}}"] & & \mathcal{L} \arrow[u] \arrow[loop below, "\mathbf{CCr}"]
\end{tikzcd}
\qquad\qquad Q
$$

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Linear-Cartesian Theories

$$
\begin{tikzcd}
\mathbb{F}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[d, shift right=1.5ex, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{F} \arrow[d, shift right=3ex, "\iota_!"'] \arrow[d, phantom, shift right=1.5ex, "\dashv"] \arrow[d, phantom, shift left=1.5ex, "\dashv"] \arrow[d, shift left=3ex, "s^*"] \\
\mathbb{L}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[u, shift right=1.5ex, "s^{\mathrm{op}}"'] \arrow[loop below, "S^{\mathrm{op}}"] & \mathcal{L} \arrow[u, "s_!"] \arrow[loop below, "\mathbf{CCr}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
s_!(Q) \\
Q \arrow[u, mapsto]
\end{tikzcd}
$$

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Linear-Cartesian Theories

$$
\begin{tikzcd}
\mathbb{F}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[d, shift right=3, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{F} \arrow[d, shift right=6, "\iota_!"'] \arrow[d, shift left=6, "s^*"] \arrow[d, phantom, "\dashv s_! \dashv"] \\
\mathbb{L}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[u, shift right=3, "s^{\mathrm{op}}"'] \arrow[loop below, "S^{\mathrm{op}}"] & \mathcal{L} \arrow[u] \arrow[loop below, "\mathbf{CCr}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& s_!(Q) \\
\mathbf{CCr}(Q) \arrow[from=ur, mapsto] & Q \arrow[u, mapsto]
\end{tikzcd}
$$

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Linear-Cartesian Theories

$$
\begin{tikzcd}
\mathbb{F}^{\mathrm{op}} \arrow[rr, "\mathcal{Y}"] \arrow[d, shift right=2, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] & & \mathcal{F} \arrow[d, shift right=4, "\iota_!"'] \arrow[d, shift left=4, "s^*"] \arrow[d, phantom, "\dashv s_! \dashv"] \\
\mathbb{L}^{\mathrm{op}} \arrow[rr, "\mathcal{Y}"] \arrow[u, shift right=2, "s^{\mathrm{op}}"'] \arrow[loop below, "S^{\mathrm{op}}"] & & \mathcal{L} \arrow[u] \arrow[loop below, "\mathbf{CCr}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & s_!(Q) \arrow[dl, mapsto] & \\
\mathbf{CCr}(Q) & & Q \arrow[ul, mapsto]
\end{tikzcd}
$$

Cartesian Core: $\mathbf{CCr}(Q)(\ell, c) = \begin{cases} Q(0, c) & \ell = 0 \\ \emptyset & \text{otherwise} \end{cases}$

$\mathbf{CCr}(Q)$ is a commutative comonoid

Linear-Cartesian Theories

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{F}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[d, shift right=2, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{F} \arrow[d, shift right=4, "\iota_!"'] \arrow[d, shift left=4, "s^*"] \arrow[d, phantom, "\dashv s_! \dashv"] \\
\mathbb{L}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[u, shift right=2, "s^{\mathrm{op}}"'] \arrow[loop below, "S^{\mathrm{op}}"] & \mathcal{L} \arrow[u] \arrow[loop below, "\mathbf{CCr}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& s_!(Q) \arrow[dl, mapsto] \\
\mathbf{CCr}(Q) & Q \arrow[u, mapsto]
\end{tikzcd}
$$

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The counit $\mathbf{CCr}(Q) \xrightarrow{\varepsilon} Q$ is a comodel of $\mathfrak{L}$

Linear-Cartesian Theories

$$
\begin{tikzcd}
\mathbb{F}^{\mathrm{op}} \arrow[r, "\mathcal{Y}"] \arrow[d, shift right=2, "\iota^{\mathrm{op}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{F} \arrow[d, shift right=4, "\iota_!"'] \arrow[d, shift left=4, "s^*"] \arrow[d, phantom, "\dashv s_! \dashv"] \\
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& s_!(Q) \arrow[dl, mapsto] \\
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$$
\begin{array}{ccccc}
\mathcal{L} & \hookrightarrow & \mathrm{coMod}(\mathfrak{L}, \mathcal{L}) & \xrightarrow{\cong} & \mathrm{SM}(\mathbb{L}^{\mathrm{op}}, \mathcal{L}) \\
Q & \longmapsto & (\ \varepsilon : \mathbf{CCr}(Q) \to Q\ ) & \longmapsto & Q^{\otimes -}
\end{array}
$$

Full Substructural Theories

Full Substructural Theories

Equational Presentation: $\mathfrak{M}$

$$
\begin{tikzcd}
I \arrow[rr] \arrow[dr] & & C & C, C \arrow[l] & \\
& A \arrow[ur] & & R \arrow[ul] & R, R \arrow[l] \arrow[ul] \\
& & L \arrow[ul] \arrow[uu] \arrow[ur] & &
\end{tikzcd}
$$

Full Substructural Theories

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& A \arrow[ur] & & R \arrow[ul] & R, R \arrow[l] \arrow[ul] \\
& & L \arrow[ul] \arrow[uu] \arrow[ur] & &
\end{tikzcd}$$

C commutative monoid
A pointed object
R commutative semigroup
Coercions commute and respect operations

Full Substructural Theories

Theory:

$$
\begin{tikzcd}
 & \mathbb{M} \arrow[dl, shift right=1ex] \arrow[dr, shift left=1ex] \arrow[dd, shift right=1ex] \arrow[dd, phantom, "\dashv"] \arrow[dl, phantom, "\perp" sloped] \arrow[dr, phantom, "\top" sloped] & \\
\mathbb{A} \arrow[ur, shift right=1ex] \arrow[dr, shift right=1ex] & & \mathbb{R} \arrow[ul, shift left=1ex] \arrow[dl, shift left=1ex] \\
 & \mathbb{F} \arrow[ul, shift right=1ex] \arrow[ur, shift left=1ex] \arrow[uu, shift right=1ex] \arrow[ul, phantom, "\top" sloped] \arrow[ur, phantom, "\perp" sloped] &
\end{tikzcd}
$$

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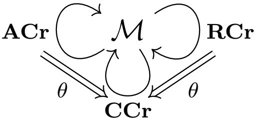
Theory:

$$
\begin{tikzcd}
 & \mathbb{M} \arrow[dl, shift right] \arrow[dd, shift right] \arrow[dr, shift left] & \\
\mathbb{A} \arrow[ur, shift right] \arrow[dr, shift right] \arrow[ur, phantom, "\perp"] \arrow[dr, phantom, "\top"] & \arrow[phantom, "\dashv"] & \mathbb{R} \arrow[ul, shift left] \arrow[dl, shift left] \arrow[ul, phantom, "\top"] \arrow[dl, phantom, "\perp"] \\
 & \mathbb{F} \arrow[ul, shift right] \arrow[uu, shift right] \arrow[ur, shift right] &
\end{tikzcd}
$$

Presheaves:



Full Substructural Theories

Theory:

$$
\begin{tikzcd}
& \mathbb{M} \arrow[dl, shift right] \arrow[dd, shift right] \arrow[dr, shift left] & \\
\mathbb{A} \arrow[ur, shift right] \arrow[ur, phantom, "\dashv"] \arrow[dr, shift left] \arrow[dr, phantom, "\top"] & & \mathbb{R} \arrow[ul, shift left] \arrow[ul, phantom, "\top"] \arrow[dl, shift left] \arrow[dl, phantom, "\dashv"'] \\
& \mathbb{F} \arrow[ul, shift left] \arrow[uu, shift right] \arrow[uu, phantom, "\dashv"] \arrow[ur, shift right] &
\end{tikzcd}
$$

Presheaves:

$$
\begin{tikzcd}
\mathbf{ACr} \arrow[dr, Rightarrow, "\theta"'] & \mathcal{M} \arrow[loop left, "\mathbf{ACr}"] \arrow[loop right, "\mathbf{RCr}"] \arrow[loop below, "\mathbf{CCr}"] & \mathbf{RCr} \arrow[dl, Rightarrow, "\theta"] \\
& \mathbf{CCr} &
\end{tikzcd}
$$

coModel:

$$
\begin{tikzcd}
& \mathbf{CCr}(Q) \arrow[dl, "\theta"'] \arrow[dr, "\theta"] \arrow[dd, "\varepsilon"] & \\
\mathbf{ACr}(Q) \arrow[dr, "\varepsilon"'] & & \mathbf{RCr}(Q) \arrow[dl, "\varepsilon"] \\
& Q &
\end{tikzcd}
$$

Full Substructural Theories

Theory:

$$
\begin{tikzcd}
 & \mathbb{M} \arrow[dl, shift right] \arrow[d, shift right] \arrow[dr, shift left] & \\
\mathbb{A} \arrow[ur, shift right, phantom, "\perp"] \arrow[ur, shift right] \arrow[dr, shift left] & \arrow[l, phantom] & \mathbb{R} \arrow[ul, shift left] \arrow[dl, shift left] \\
 & \mathbb{F} \arrow[ul, shift left] \arrow[u, shift right, "\dashv"'] \arrow[ur, shift left] &
\end{tikzcd}
$$

Presheaves:

$$
\begin{tikzcd}
\mathbf{ACr} \arrow[dr, Rightarrow, "\theta"'] & \mathcal{M} \arrow[loop above] \arrow[loop below] & \mathbf{RCr} \arrow[dl, Rightarrow, "\theta"] \\
 & \mathbf{CCr} &
\end{tikzcd}
$$

coModel:

$$
\begin{tikzcd}
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\mathbf{ACr}(Q) \arrow[dr, "\varepsilon"'] & & \mathbf{RCr}(Q) \arrow[dl, "\varepsilon"] \\
 & Q &
\end{tikzcd}
$$

$$
\mathcal{M} \hookrightarrow \mathrm{coMod}(\mathfrak{M}, \mathcal{M}) \xrightarrow{\cong} \mathrm{SM}(\mathbb{M}^{\mathrm{op}}, \mathcal{M})
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Free-Forgetful Adjunctions

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Every inclusion of equational presentations induces a free-forgetful adjunction

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Equational Presentations:

$$\begin{tikzcd} & \mathfrak{L} & \\ \mathfrak{B} \arrow[ur] & & \mathfrak{F} \arrow[ul] \end{tikzcd}$$

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Theories:

$$\begin{tikzcd}
 & \mathbb{L} \arrow[dr, bend left] \arrow[dr, phantom, "\perp"] & \\
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$$

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$$
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 & \mathcal{L} \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathcal{B} \arrow[ur, shift left, "\perp"'] \arrow[rr, shift left, "\perp"'] & & \mathcal{F} \arrow[ll, shift left] \arrow[ul, shift left, "\top"']
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$$

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Presheaves:

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\end{tikzcd}$$

Monoids:

$$\begin{tikzcd}
& \mathbf{LCOp} \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathbf{SymOp} \arrow[ur, shift left] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left] \arrow[rr, phantom, "\perp"] & & \mathbf{Law} \arrow[ul, shift left] \arrow[ul, phantom, "\top"] \arrow[ll, shift left]
\end{tikzcd}$$

Other Aspects of the Work

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Fiore-Gambino-Hyland-Winskel (2008):

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We have bicategories for all these settings

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This does not work for arbitrary symmetric monoidal equational presentations, but...

Sorts + Coercions = join semi-lattice

Coercions respect operations

eg. Total order on $\mathbb{N} \rightsquigarrow$ Graded Operads

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Single-variable substitution for linear, affine, relevant and cartesian settings has been developed

Work-in-progress: single-variable substitution for other settings